The equivariant de Rham complex on a simplicial $G_*$-manifold

Naoya Suzuki

Abstract

We show that when a simplicial Lie group acts on a simplicial manifold $\{X_\ast\}$, we can construct a bisimplicial manifold and the de Rham complex on it. This complex is quasi-isomorphic to the equivariant simplicial de Rham complex on $\{X_\ast\}$ and its cohomology group is isomorphic to the cohomology group of the fat realization of the bisimplicial manifold. We also exhibit a cocycle in the equivariant simplicial de Rham complex.

1 Introduction

Simplicial manifold is a sequence of manifolds together with face and degeneracy operators satisfying some relations. There is a well-known way to construct the de Rham complex on a simplicial manifold (see [2][5][9], for instance). In [8], Meinrenken introduced the equivariant version of the de Rham complex on a simplicial manifold. That is a double complex whose components are equivariant differential forms which is called the Cartan model [1]. This complex is a generalization of Weinstein’s one in [14]. In this paper, we show that when a simplicial Lie group acts on a simplicial manifold $\{X_\ast\}$, we can construct a bisimplicial manifold and explain that the de Rham complex on it is quasi-isomorphic to the equivariant de Rham complex on $\{X_\ast\}$. We explain also that its cohomology group is isomorphic to the cohomology
group of the fat realization of the bisimplicial manifold. At the last section, we exhibit a cocycle in the equivariant de Rham complex on a simplicial manifold $NSO(4)$.

2 Review of the simplicial de Rham complex

2.1 Simplicial manifold

Definition 2.1 ([10]). Simplicial manifold is a sequence of manifolds $X = \{X_q; (q = 0, 1, 2 \cdots)\}$ together with face operators $\varepsilon_i : X_q \rightarrow X_{q-1}$ ($i = 0, 1, 2 \cdots q$) and degeneracy operator $\eta_i : X_q \rightarrow X_{q+1}$ ($i = 0, 1, 2 \cdots q$) which are all smooth maps and satisfy the following identities:

\[
\varepsilon_i\varepsilon_j = \varepsilon_{j-1}\varepsilon_i \quad i < j \\
\eta_i\eta_j = \eta_{j+1}\eta_i \quad i \leq j \\
\varepsilon_i\eta_j = \begin{cases} 
\eta_{j-1}\varepsilon_i & i < j \\
id & i = j, \ i = j + 1 \\
\eta_j\varepsilon_{i-1} & i > j + 1.
\end{cases}
\]

Simplicial Lie group $\{G_*\}$ is a simplicial manifold such that all $G_n$ are Lie groups and all face and degeneracy operators are group homomorphisms.

For any Lie group $G$, we have simplicial manifolds $NG$, $PG$ and simplicial $G$-bundle $\gamma : PG \rightarrow NG$ as follows:

\[
NG(q) = \underbrace{G \times \cdots \times G}_q \ni (g_1, \cdots, g_q) : \\
\text{face operators} \quad \varepsilon_i : NG(q) \rightarrow NG(q-1) \\
\varepsilon_i(g_1, \cdots, g_q) = \begin{cases} 
(g_2, \cdots, g_q) & i = 0 \\
(g_1, \cdots, g_i, g_{i+1}, \cdots, g_q) & i = 1, \cdots, q-1 \\
(g_1, \cdots, g_{q-1}) & i = q
\end{cases}
\]

\[
PG(q) = \underbrace{\bar{G} \times \cdots \times \bar{G}}_{q+1} \ni (\bar{g}_1, \cdots, \bar{g}_{q+1}) .
\]
face operators \( \bar{\varepsilon}_i : PG(q) \to PG(q - 1) \)

\[
\bar{\varepsilon}_i(\bar{g}_1, \ldots, \bar{g}_{q+1}) = (\bar{g}_1, \ldots, \bar{g}_i, \bar{g}_{i+2}, \ldots, \bar{g}_{q+1}) \quad i = 0, 1, \ldots, q
\]

Degeneracy operators are also defined but we do not need them here.

We define \( \gamma : PG \to NG \) as \( \gamma(\bar{g}_1, \ldots, \bar{g}_{q+1}) = (\bar{g}_1 \bar{g}_2^{-1}, \ldots, \bar{g}_q \bar{g}_{q+1}^{-1}) \).

For any simplicial manifold \( \{X_\ast \} \), we can associate a topological space \( \| X_\ast \| \) called the fat realization defined as follows:

\[
\| X_\ast \| := \coprod_n \Delta^n \times X_n / (\varepsilon^i t, x) \sim (t, \varepsilon^i x).
\]

Here \( \Delta^n \) is the standard \( n \)-simplex and \( \varepsilon^i \) is a face map of it. It is well-known that \( \| \gamma \| : \| PG \| \to \| NG \| \) is the universal bundle \( EG \to BG \) (see [5] [9] [10], for instance).

### 2.2 The double complex on a simplicial manifold

**Definition 2.2.** For any simplicial manifold \( \{X_\ast \} \) with face operators \( \{\varepsilon_\ast \} \), we have a double complex \( \Omega^{p,q}(X_\ast) := \Omega^q(X_p) \) with derivatives defined as follows:

\[
d' := \sum_{i=0}^{p+1} (-1)^i \varepsilon^i, \quad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).
\]

For any simplicial manifold the following holds.

**Theorem 2.1** ([2] [5] [9]). There exist a ring isomorphism

\[
H^*(\Omega^*(X_\ast)) \cong H^*(\| X_\ast \|).
\]

Here \( \Omega^*(X_\ast) \) means the total complex.

\( \square \)
3 Simplicial $G_\ast$-manifold

Let $\{X_\ast\}$ be a simplicial manifold and $\{G_\ast\}$ be a simplicial Lie group which acts on $\{X_\ast\}$ by left, i.e. $G_n$ acts on $X_n$ by left and this action is commutative with face and degeneracy operators of $\{X_\ast\}$. We call $\{X_\ast\}$ a simplicial $G_\ast$-manifold.

A bisimplicial manifold is a sequence of manifolds with horizontal and vertical face operators which commute with each other.

Given a simplicial $G_\ast$-manifold $\{X_\ast\}$, we can construct a bisimplicial manifold $\{X_\ast \rtimes NG_\ast(\ast)\}$ in the following way:

$$X_p \rtimes NG_p(q) := X_p \times G_p \times \cdots \times G_p.$$

Horizontal face operators $\varepsilon^{Ho}_i: X_p \rtimes NG_p(q) \to X_{p-1} \rtimes NG_{p-1}(q)$ are the same as the face operators of $X_p$ and $G_p$. Vertical face operators $\varepsilon^{Ve}_i: X_p \rtimes NG_p(q) \to X_p \rtimes NG_p(q-1)$ are

$$\varepsilon^{Ve}_i(x, g_1, \cdots, g_q) = \begin{cases} (x, g_2, \cdots, g_q) & i = 0 \\ (x, g_1, \cdots, g_{i+1}, \cdots, g_q) & i = 1, \cdots, q - 1 \\ (g_q x, g_1, \cdots, g_{q-1}) & i = q. \end{cases}$$

Example 3.1. Suppose $G_n = H$ is a compact subgroup of $G$ and $H$ acts on $NG(n)$ as follows:

$$h \cdot (g_1, g_2, \cdots, g_n) = (hg_1h^{-1}, hg_2h^{-1}, \cdots, hg_nh^{-1}).$$

Then $X_n = NG(n)$ is a simplicial $H$-manifold and $\| NG(\ast) \rtimes NH(\ast) \|$ is $B(G \rtimes H)$ ([11]).

Example 3.2. $PG(n)$ acts on $PG(n)$ itself by left as follows:

$$(\bar{k}_1, \cdots, \bar{k}_{n+1}) \cdot (\bar{g}_1, \cdots, \bar{g}_{n+1}) = (\bar{k}_1\bar{g}_1\bar{k}_1^{-1}, \cdots, \bar{k}_{n+1}\bar{g}_{n+1}\bar{k}_{n+1}^{-1}).$$

So $PG(\ast)$ is a simplicial $PG(\ast)$-manifold ([8]). If $G$ is compact, $\| PG(\ast) \rtimes N(PG(\ast))(\ast) \|$ is a fat realization of a simplicial space $PG(n) \times_{PG(n)} EPG(n)$.

Example 3.3. If the action of $\{G_\ast\}$ on $\{X_\ast\}$ is trivial, $\| X_\ast \rtimes NG_\ast(\ast) \|$ is $\| X_\ast \|$.
Example 3.4. Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a $G$-groupoid, i.e. both $\Gamma_1$ and $\Gamma_0$ are $G$-manifolds and all structure maps are $G$-equivariant. We define a simplicial manifold $N\Gamma$ as follows:

\[ N\Gamma(p) := \{(x_1, \cdots, x_p) \in \prod_{j=1}^{p-1} \Gamma_1 \mid t(x_j) = s(x_{j+1}) \} \]

face operators $\varepsilon_i : N\Gamma(p) \to N\Gamma(p-1)$

\[ \varepsilon_i(x_1, \cdots, x_p) = \begin{cases} (x_2, \cdots, x_p) & i = 0 \\ (x_1, \cdots, m(x_i, x_{i+1}), \cdots, x_p) & i = 1, \cdots, p-1 \\ (x_1, \cdots, x_{p-1}) & i = p. \end{cases} \]

Here $s, t, m$ mean the source and target maps, and the multiplication ([13]). Then $N\Gamma(*)$ is a simplicial $G$-manifold.

4 The equivariant simplicial de Rham complex

4.1 The triple complex

Definition 4.1. For a bisimplicial manifold $\{X_{s,*}\}$, we can construct a triple complex on it in the following way:

\[ \Omega^{p,q,r}(X_{s,*}) := \Omega^r(X_{p,q}) \]

Derivatives are:

\[ d' := \sum_{i=0}^{p+1} (-1)^i (\varepsilon_i^{H^o})^* \quad d'' := \sum_{i=0}^{q+1} (-1)^i (\varepsilon_i^{V^c})^* \times (-1)^p \]

\[ d''' := (-1)^{p+q} \times \text{the exterior differential on } \Omega^*(X_{p,q}). \]

Repeating the same argument in [11], we obtain the following theorem.

Theorem 4.1. There exists an isomorphism

\[ H(\Omega^*(X_s \times NG_s(*))) \cong H^*([X_s \times NG_s(*)]). \]

Here $\Omega^*(X_s \times NG_s(*))$ means the total complex.
4.2 The equivariant simplicial de Rham complex

When a compact Lie group $G$ acts on a manifold $M$, there is the complex of equivariant differential forms $\Omega^*_G(M) := (\Omega^*(M) \otimes S(G^*))^G$ with the differential $d_G$ defined by $(d_G \alpha)(X) := (d - i_{X_M})(\alpha(X))$ ([1] [3]). Here $G$ is the Lie algebra of $G$, $S(G^*)$ is the algebra of polynomial functions on $G$, $\alpha \in \Omega^*_G(M), X \in G$ and $X_M$ denote the vector field on $M$ generated by $X$. This is called the Cartan Model. We can define the double complex $\Omega^*_G(X_*)$ in the same way as in Definition 2.2. This double complex is originally introduced by Meinrenken in [8].

Again, repeating the same argument in [11], we obtain the following theorem.

**Theorem 4.2.** If every $G_n$ is compact, there exists an isomorphism

$$H(\Omega^*_G(X_*)) \cong H(\Omega^*(X_\ast \times NG_\ast(*))).$$

Here $\Omega^*_G(X_*)$ means the total complex.

Remark 4.1. In the case that $G_n$ is not compact, we need to use “the Getzler model” of the equivariant cohomology in [6].

4.3 Cocycle in the equivariant simplicial de Rham complex

In this section we take $G = SO(4)$ and construct a cocycle in $\Omega^4(\text{NSO}(4))$, whose cohomology is isomorphic to $H^*(\text{BSO}(4) \times \text{SO}(4))$.

Recall that there is a cocycle in $\Omega^4(\text{NSO}(4))$ described in the following way.

**Theorem 4.3 ([12]),** The cocycle which represents the Euler class of $\text{ESO}(4) \rightarrow \text{BSO}(4)$ in $\Omega^4(\text{NSO}(4))$ is the sum of the following $E_{1,3}$ and $E_{2,2}$:

$$\begin{array}{cccc}
0 & \uparrow d & \Omega^3(\text{SO}(4) \times \text{SO}(4)) & \uparrow d \\
E_{1,3} & \in & \Omega^3(\text{SO}(4)) & \xrightarrow{d'} \Omega^3(\text{SO}(4) \times \text{SO}(4)) \\
\end{array}$$

$$\begin{array}{cccc}
\uparrow -d & \rightarrow & \rightarrow & \rightarrow \\
E_{2,2} & \in & \Omega^2(\text{SO}(4) \times \text{SO}(4)) & \xrightarrow{d'} 0 \\
\end{array}$$
\[ E_{1,3} = \frac{1}{192\pi^2} \sum_{\tau \in \mathfrak{S}_4} \text{sgn}(\tau) \left( (h^{-1} dh)_{\tau(1)}(h^{-1} dh)^2_{\tau(3)\tau(4)} + (h^{-1} dh)_{\tau(3)\tau(4)}(h^{-1} dh)^2_{\tau(1)\tau(2)} \right) \]

\[ E_{2,2} = \frac{-1}{64\pi^2} \sum_{\tau \in \mathfrak{S}_4} \text{sgn}(\tau) \left( (h^{-1} dh_1)_{\tau(1)\tau(2)}(dh_2 h_2^{-1})_{\tau(3)\tau(4)} + (h^{-1} dh_1)_{\tau(3)\tau(4)}(dh_2 h_2^{-1})_{\tau(1)\tau(2)} \right). \]

**Errata 1.** In [12], there are some mistakes. Some numbers of propositions and theorems are wrong. For example, “Proposition 3.1” in P.38 should be modified as “Proposition 2.1”. The cocycle in Theorem 2.2 should be written as above. Also, \[ \frac{\partial^2}{\partial y_1 \partial y_2} b(\gamma_1, \gamma_2) \]_{y=0} and \( \alpha(\xi_1, \xi_2) \) should be written as follows.

\[ \left[ \frac{\partial^2}{\partial y_1 \partial y_2} b(\gamma_1, \gamma_2) \right]_{y=0} = \frac{-1}{64\pi^2} \sum_{\tau \in \mathfrak{S}_4} \text{sgn}(\tau) \int_0^1 \left( \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(1)\tau(2)} \xi_2(\theta)_{\tau(3)\tau(4)} + \left( \frac{\partial \xi_2(\theta)}{\partial \theta} \right)_{\tau(3)\tau(4)} \xi_1(\theta)_{\tau(1)\tau(2)} \right) d\theta. \]

\[ \alpha(\xi_1, \xi_2) := \frac{-1}{64\pi^2} \sum_{\tau \in \mathfrak{S}_4} \left( \text{sgn}(\tau) \cdot \right. \]

\[ \int_0^1 \left( \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(1)\tau(2)} \xi_2(\theta)_{\tau(3)\tau(4)} + \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(3)\tau(4)} \xi_2(\theta)_{\tau(1)\tau(2)} \right) \]

\[ - \left( \frac{\partial \xi_2(\theta)}{\partial \theta} \right)_{\tau(1)\tau(2)} \xi_1(\theta)_{\tau(3)\tau(4)} - \left( \frac{\partial \xi_2(\theta)}{\partial \theta} \right)_{\tau(3)\tau(4)} \xi_1(\theta)_{\tau(1)\tau(2)} \right) d\theta. \]

\[ \square \]

Now following Jeffrey and Weinstein’s idea, we construct a cocycle in \( \Omega_{SO(4)}^1(NSO(4)) \).

We take a cochain \( \mu \in (\Omega^1(G) \otimes \mathcal{G}^*)^G \) as follows:

\[ \mu(X) = \frac{-1}{64\pi^2} \sum_{\tau \in \mathfrak{S}_4} \text{sgn}(\tau) \left( (X)_{\tau(1)\tau(2)}(h^{-1} dh)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(h^{-1} dh)_{\tau(1)\tau(2)} \right) \]
Lemma 4.1. \( i_{X_g} E_{1,3} = d\mu(X) \)

Proof. Since \( i_X (g^{-1}dg) = i_X (dgg^{-1}) = X \), the following equation holds.

\[
i_{X_g} E_{1,3} = i_{X - X} E_{1,3} = i_X E_{1,3} - i_X E_{1,3}
\]

\[
= \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) \left( (X)_{\tau(1)\tau(2)} (h^{-1} dh_{\tau(3)\tau(4)}) + (X)_{\tau(3)\tau(4)} (h^{-1} dh_{\tau(1)\tau(2)}) \right)
\]

\[
- \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) \left( (X)_{\tau(1)\tau(2)} (dh h^{-1})_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)} (dh h^{-1})_{\tau(1)\tau(2)} \right)
\]

\[
= d\mu(X).
\]

Lemma 4.2. \( i_{X_{g \times g}} E_{2,2} = (\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*) \mu(X) \)

Proof. \( (\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*) \mu(X) \)

\[
= \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) \left( (X)_{\tau(1)\tau(2)} (h^{-1} dh_{\tau(3)\tau(4)}) + (X)_{\tau(3)\tau(4)} (h^{-1} dh_{\tau(1)\tau(2)}) \right)
\]

\[
- \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) \left( (X)_{\tau(1)\tau(2)} (dh h^{-1})_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)} (dh h^{-1})_{\tau(1)\tau(2)} \right)
\]

\[
+ \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) \left( (X)_{\tau(1)\tau(2)} (h^{-1} h_1^{-1} dh_1 h_2 + h_2^{-1} dh_2)_{\tau(3)\tau(4)} \right)
\]

\[
+ (X)_{\tau(3)\tau(4)} (h_2^{-1} h_1^{-1} dh_1 h_2 + h_2^{-1} dh_2)_{\tau(1)\tau(2)}
\]

\[
+ \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) \left( (X)_{\tau(1)\tau(2)} (dh_1 h_1^{-1} + h_1 dh_2 h_2^{-1} h_1^{-1})_{\tau(3)\tau(4)} \right)
\]

\[
+ (X)_{\tau(3)\tau(4)} (dh_1 h_1^{-1} + h_1 dh_2 h_2^{-1} h_1^{-1})_{\tau(1)\tau(2)}
\]

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\[- \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(h^{-1}_1dh_1)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(h^{-1}_1dh_1)_{\tau(1)\tau(2)})\]

\[- \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(dh_1h^{-1}_1)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(dh_1h^{-1}_1)_{\tau(1)\tau(2)})\]

\[= - \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(dh_2h^{-1}_2)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(dh_2h^{-1}_2)_{\tau(1)\tau(2)})\]

\[+ \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(h^{-1}_2h^{-1}_1dh_1h_2)_{\tau(3)\tau(4)}\]

\[+ (X)_{\tau(3)\tau(4)}(h^{-1}_2h^{-1}_1dh_1h_2)_{\tau(1)\tau(2)})\]

\[+ \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(h_1dh_2h^{-1}_2h^{-1}_1)_{\tau(3)\tau(4)}\]

\[+ (X)_{\tau(3)\tau(4)}(h_1dh_2h^{-1}_2h^{-1}_1)_{\tau(1)\tau(2)})\]

\[- \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(h^{-1}_1dh_1)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(h^{-1}_1dh_1)_{\tau(1)\tau(2)})\]

\[= i_{X_1}E_{2,2} + i_{X_2}E_{2,2} - i_{\bar{X}_1}E_{2,2} - i_{\bar{X}_2}E_{2,2} = i_{X_1-\bar{X}_1+X_2-\bar{X}_2}E_{2,2} = i_{X_G\bar{X}_G}E_{2,2}.\]

**Lemma 4.3.** \(-i_{X_G}\mu(X) = 0\)

**Proof.** \(-i_{X_G}\mu(X) = -i_{X-\bar{X}}\mu(X)\)

\[= \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(X)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(X)_{\tau(1)\tau(2)})\]

\[+ \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((h^{-1}Xh)_{\tau(1)\tau(2)}(X)_{\tau(3)\tau(4)} + (h^{-1}Xh)_{\tau(3)\tau(4)}(X)_{\tau(1)\tau(2)})\]

\[- \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((hXh^{-1})_{\tau(1)\tau(2)}(X)_{\tau(3)\tau(4)} + (hXh^{-1})_{\tau(3)\tau(4)}(X)_{\tau(1)\tau(2)})\]

\[- \frac{1}{64\pi^2} \sum_{\tau \in \Theta} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(X)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(X)_{\tau(1)\tau(2)}) = 0.\]
As a result, we obtain the following theorem.

**Theorem 4.4.** $E_{1,3} + E_{2,2} + \mu$ is a cocycle in $\Omega^4_{SO(4)}(NSO(4))$.

**References**


e-mail: nysuzuki@akita-nct.ac.jp